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SOLVING MICROECONOMIC MODEL USING METHODS OF FUNCTIONAL ANALYSIS

Abstract: *Models in microeconomics are usually created in order to achieve corporate goals such as boosting the performance, efficiency, or to secure a company's survival in the time of crisis. This paper presents a nonlinear dynamical model of production, warehousing, and sale of fast moving consumer goods, while respecting the effect of historical values on current changes. The model can be used for planning production, or as part of models dealing with cooperation or competition. The solution is demonstrated on a practical example and is presented graphically.*

Keywords: *goods, delay differential equations, functional differential equation, microeconomic model*

JEL Classification: C02, C69

1. Introduction

Nowadays, increasingly more emphasis is put on boosting the efficiency of management processes, productivity of work, and optimization of all the other activities going on in a company. Thanks to modelling economic processes, a clear picture of processes and relationships in the modelled area can be obtained, and the information can be used to maintain a stable position on the market, to increase the competitiveness of a company, or to reduce competition risks. A formal description of the model requires a clear, succinct, and illustrative depiction of reality, often revealing inconsistent behavior of the process monitored. However, increasingly more often, when modelling in economics, we encounter rather difficult practical problems for which the description using ordinary differential equations is not sufficient. They are primarily models whose development may depend not only on the present, but also on the past state, and they may successfully be described by means of functional analysis.

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The paper presents a solution to a nonlinear dynamical model of the production, warehousing and sale of fast moving consumer goods assuming that reactions of some parts of the system may be delayed.Such a model may then be used for planning production, or it may be part of more sophisticated economic models. The scientific objective is to verify the solvability of such a problem, to which end the methods of analysis and synthesis, and methods of mathematical analysis (methods of solution of functional differential equations) have been employed. The model in the paper is illustrated by a specific problem including a graphical interpretation of the solution.

2. Literature review

A microeconomic mathematical model may help improve the management of a particular production unit. Dynamical models, described by differential equations, represent one of the ways to describe the behaviour of a microeconomic system, whose dynamical properties play an important role.Dynamical pricing models and their potential extensions are discussed e.g. in(Rana and Olivier, 2014). Studies (Gallego and Hu, 2014)address dynamical pricing of perishable goods.In publication(Baumol and Blinder, 2015) authors look into the consumer choice theory, and the maximization of utility in case of budget constraints.(Sauquet et al., 2011)deal with exogenous price elasticity of demand.(Zhang et al.,2014) and(Chen and Hu, 2012) model the function of deterministic demand in a supply chain, while monitoring price effects. They work on the assumption of balanced prices and study in detail the profit sensitivity to changes in various factors. Competitive relationships in a supply chain are analysed in(Chung et al., 2014). The robust optimization model in which demand is regarded as a decreasing function is described in (Lim, 2013). Duopoly model seen from the point of view of the game theory is expounded in a paper (Shulman and Geng,2012), where they consider two companies and heterogeneous consumers. There are two publications on air transport based on an empirical study. In work (Brueckner et al., 2015) authors analysed the effect of luggage fees on flight ticket prices in the air transport.(Nicolae et al., 2013) deal with the issue of delayed flights in connection with the number of checked-in luggage.

One of the models emphasised by contemporary scientific studies, is a product-inventory model, which covers production as well as the inventory of finished goods. The traditional approach to the solution to this problem, in terms of optimization, is described in (Sethi and Thompson, 2000), while (Ortega and Lin, 2004) focus on the application of management theories in the field of product-inventory models. This field has seen numerous papers published in the past years, such as publication (Teng and Chang, 2005) on the application of models of stock-dependent demand rate. Another sphere of interest is represented by papers on inventory optimization, e.g.(Bakker et al., 2012).

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Further example can be (Tokarev, 2002), who proposed a microeconomic model of short-term crediting and debt repayment for a small firm, or a model (Grigoriev and Khailov, 2005), which is supplemented with an additional condition.

The model describing the production process and the sale of goods using the system of nonlinear equations is described in paper (Grigorieva and Khailov, 2005), who based their work on principles specified in (Lancaster, 2012) when drawing up their model. Papers expound possible solutions from the point of the theory of optimal management. Application of Pontryagin's principle for solving a similar problem is described by Grigorieva andKhailovin publication (Grigorieva and Khailov,2015). (Gorskij et all., 1993) demonstrate a nonlinear mathematical models of a microeconomic system which produces and sells fast moving consumer goods.The paper compares the model with a statistical model.

3. Framing a problem

Let us have a nonlinear mathematical model of a microeconomic system which deals with the production, warehousing and sale of consumer goods. The model is based on works by (Grigorieva and Khailov, 2005), and(Gorskijand Lokshin, 2002), which describe its various variants in detail and discuss solutions to such models. The model can be described by a nonlinear system of differential equations:

$$
\begin{gathered}
x_1'(t) = u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) - k_1 x_1(t) \\
x_2'(t) = n(p)\big(Y - x_2(t)\big)x_1(t) + k_2 x_2(\mathrm{t}) \\
x_1(0) = x_1^0 \geq 0, \qquad x_2(0) = x_2^0 \geq 0, \;\; t \in [0, T]
\end{gathered} \tag{1}
$$

x_1– the number of goods on the market

x_2 – customers' reserves

$u(t)$ – the rate of production

k_1 – rate of goods damage($k_1 x_1(t)$became defective per time unit)

k_2 – the consumption rate

p – the selling price

n – the coefficient of the rate of sale of perishable goods n(p)=n0/p, n0 is an invariable

Y – the potential demand

When examining the dependence of some economic quantities in time, we need to take into consideration the fact that a quantity is dependent on its previous values, or on previous values of other quantities. In other words, the effects of time delay in this process must be allowed for.

Let us now admit that changes caused by gradual consumption on the customer's side and damage on the vendor's side may manifest themselves with a

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certain delay, which means that the system is affected by past changes. If such changes are admitted and feedback is added to the system, a new system will appear:

$$\begin{aligned} &x_1'(t) = u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) + k_1 x_1(t - \Delta_1) \\ &x_2'(t) = n(p)\big(Y - x_2(t)\big)x_1(t) + k_2 x_2(t - \Delta_2) \\ &x_1(0) = x_1^0 \geq 0,\ x_2(0) = x_2^0 \geq 0,\ \ t \in [0, T], \end{aligned} \qquad (2)$$

where Δ_1 and Δ_2 express the time needed to detect changes in variables x_1 and x_2,and they will still be called delays.

Given the existing knowledge of solving differential equations with delayed argument (more generally, with deviated argument), or so-called functional differential equations, we can apply methods of solution of this particular mathematical model, thus achieving a wider range of results, including comparison of effects caused by various parameters.

4. Construction of a solution

Since the second half of the 20th century new mathematical models have been appearing, describing a dynamical model by means of differential equations, which can be used to realistically describe various processes in real life (David and Křápek, 2013) or (Plaček, 2013). If this view is adopted, time needs to be understood as a continuous quantity, which allows us to employ a sophisticated mathematical apparatus of the differential and integral calculus. The result is not an estimate of parameters of pre-defined types of functions, but a function itself, whose shape attests to the properties of the quantities examined.

In case of solution to a functional differential equation or system of such equations, the solutions do not need to only depend on values in time t, but also on others (e.g. on the delay preceding time t), thus the solving process cannot be restricted solely to local methods. In such cases and in order to extend deliberations to as wide a range of practical problems as possible, employing a Caratheodory solution seems to serve the purpose, which is a more general approach to solutions dating back to the beginning of the last century. Methodological procedures employed in this study are based on this approach, i.e. the method of a priori estimates of solutions to such equations and systems, and form a natural basis for the method of gradual approximations on an appropriately affiliated operator equation with a contractive operator.

The method was illustratively applied by I. Kiguradzeto design a solution to the initial and periodic problem of a nonlinear system of ordinary differential equations (Kiguradze,1988). In cooperation with Gelashvilli he analysed in detail numerical solutions to nonlinear differential equations in (Gelashvilli and Kiguradze, 1995).Application tasks based on solving systems of differential

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equations with delay, including the description of the solution construction can be found for example in (Bobalová and Maňásek, 2006) and in cited literature therein.

The above mentioned procedures have been modified and applied in solutions to numerous specific problems, modelling, among others, the economic process examined in our paper.

The solution to our problem was designed by means of Maple system, mathematical software, the advantage of which is the possibility to find even a symbolic solution in some cases. The problem in question was solved using numerical methods, built-in in Maple, which are designed to solve ordinary differential equations. Thanks to the above mentioned modern theory of solving differential equations with delayed argument, an algorithm can be constructed leading to a solution to the model.

Let us consider the mathematical model of the economic process described in the introduction on interval $[-\Delta, T]$as follows:

$$
\begin{aligned}
x_1'(t) &= u(t) - n(p)\big(Y - x_2(t)\big)x_1(t) \\
&+ k_1\Big(\chi(t-\Delta_1)x_1(t-\Delta_1) + \big(1-\chi(t-\Delta_1)\big)x_{1_h}(t-\Delta_1)\Big) \\
x_2'(t) &= n(p)\big(Y - x_2(t)\big)x_1(t) \\
&+ k_2\Big(\chi(t-\Delta_2)x_2(t-\Delta_1) + \big(1-\chi(t-\Delta_2)\big)x_{2_h}(t-\Delta_2)\Big) \qquad (3) \\
& x_1(0) = x_{1h}(0) \geq 0,\ x_2(0) = x_{2h}(0) \geq 0,\ \ t \in [0, T],
\end{aligned}
$$

where

$$
\chi(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases},
$$

andx_{ih}, $i = 1{,}2$are functions continuous on interval $[-\Delta_i, 0]$.

A natural consequence of the requirement of continuous sequence of solutions $x_i(t)$ from interval$[0, T]$ to „historical" function$x_{ih}(t)$ ($t\epsilon[-\Delta, 0]$) is the initial condition of solution $x_i(t)$of equation (3) in the current form $x_i(t)(0) = x_{ih}(0)\ \ i = 1{,}2$.

Analogically with methods applied in the quoted studies, we shall now look for a solution to system (2) as a (direct) limit of a sequence of solutions to problems (n∈N):

$$
\begin{aligned}
x_{1\mathrm{n}}'(t) &= u(t) - n(p)\big(Y - x_{2\mathrm{n}-1}(t)\big)x_{1\mathrm{n}}(t) \\
&+ k_1\Big(\chi(t-\Delta_1)x_{1n-1}(t-\Delta_1) + \big(1-\chi(t-\Delta_1)\big)x_{1_h}(t-\Delta_1)\Big) \\
x_{2\mathrm{n}}'(t) &= n(p)\big(Y - x_{2\mathrm{n}}(t)\big)x_{1\mathrm{n}-1}(t) \\
&+ k_2\Big(\chi(t-\Delta_2)x_{2\mathrm{n}-1}(t-\Delta_1) + \big(1-\chi(t-\Delta_2)\big)x_{2_h}(t-\Delta_2)\Big) \qquad (4)
\end{aligned}
$$

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$$x_{1n}(0) = x_{1_h}(0) \geq 0,\ x_{2\mathrm{n}}(0) = x_{2_h}(0) \geq 0,\ \ t \in [0, T].$$

Given the above mentioned assumptions, each problem from the listed sequence of problems is unambiguously solvable, its solution is continuously dependent on continuous changes in initial conditions and parameters, and, at the same time, the limit of sequence of solutions converges evenly to the solution of the original problem.

5. Example – production and supplies of coffee vending machine cups

The illustrative example is based on a real situation and uses data of a production plant supplying plastic cups to vending companies, i.e. companies which run and/or sell vending machines, in particular coffee vending machines, which is a specific service sector. The current environment means for a company that it needs to cope with the pressure to reduce prices while continuously increasing running costs. It can also be observed that competition is increasingly fiercer, which results in the need to consistently strive to optimize processes.

Coffee vending companies place their vending machines in busy places and consequently goods need to be replenished several times per week; that is why it is necessary to flexibly respond to their demands for supply. On the other hand, suppliers of „hard“ components for filling up a vending machine have a low bargaining power as there are numerous prospective suppliers they can be replaced with, and at the same time, many suppliers cooperate among themselves and support one another. This paper will feature a model grasping the situation on the market and we shall study the effect of changes in historic functions and a product’s selling price, i.e. one of the model’s parameters.

In the first situation, we shall start with the available information on the sale of cups in time $t = 0$. The number of goods is given in thousands for the purpose of transparency. The demand for goods $Y = 1100$.The number of goods on the market $x_1 = 850$ and the product damage rate in warehousing is $k_1 = 0.2$. The production level is kept at $u = 650$.

Historical functions are based on particular development, monitored over the period of five weeks, and their functions were estimated as follows:level of good on the market $x_1(t) = 850 + 15 \cdot sin(0.5t)$ and the level of customers’ reserves is$x_2 = 300 + 20 \cdot sin(4t)$. The consumption rate $k_2 = 0{,}85$. Delay in market response to changes is defined as $\Delta = 3$ and customers’ response has a delay $\Delta = 2$. The selling price of goods is at the value of $p = 0.2$currency units, the coefficient of sale is $n = 14 \cdot 10^{-5}$. The unit is one week, the solution is found on interval$t = < 0; 30 >$.

In order to achieve a solution, Y approximations were needed; selected approximations for function$x_1$can be seen on figure 1.

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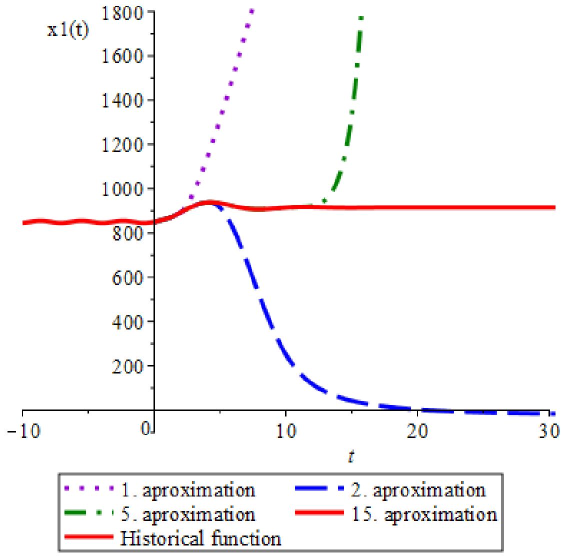

Figure 1: Selected approximations for x_1

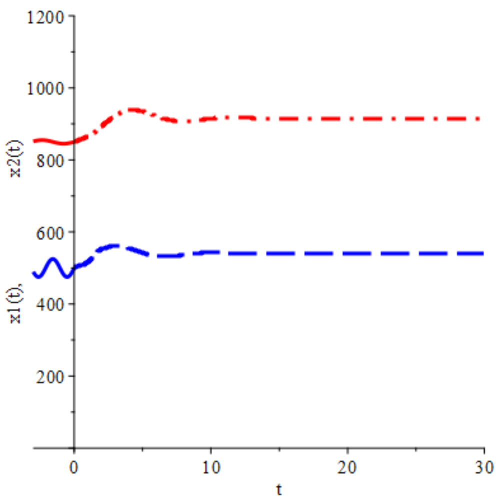

Figure 2. Model at the basic setting

As figure 2 shows, if the situation on the market remains stable and parameters are not changed, then both the number of goods on the market and the level of permanent customers' reserves stabilize in the long term. However, this situation is not realistic; for example the level of customers' reserves is excessively high; consequently, customers are expected not to accept such high reserves and are likely to lower their orders on that account. Therefore we need to consider

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situations in which some parameters of the model are not constant, but vary in time.

The following graphs demonstrate situations in which the price is not regarded as constant in the long term, and keeps changing.

Graphs on figure 3. simulate a situation in which the price oscillates around the level which is on average lower than the original price, yet it is still acceptable for the company. Allowing rather fast price fluctuations leads to an unstable situation on the market; this will require production to respond in a very flexible way. If prices change over a longer period of time, the situation on the market gradually stabilizes.

a) p=0.08+ 0.02*cos(0.02*t) **b) p =0.08+ 0.02*cos(0.6*t)**

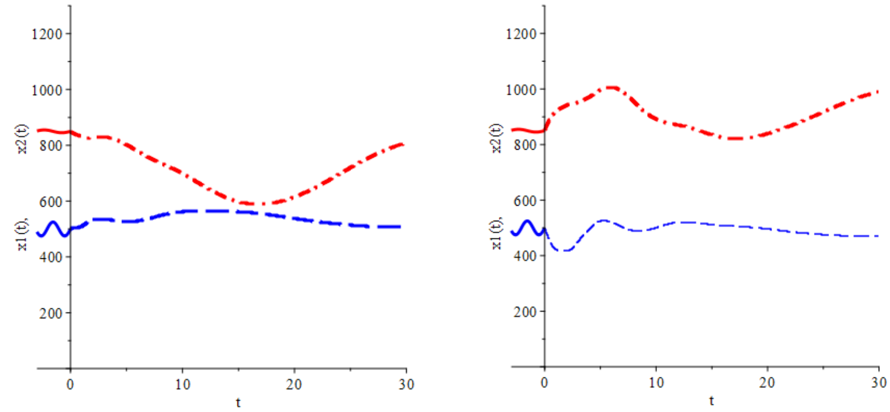

Figure 3. Fluctuating price

In the above mentioned examples the price adjustment was continuous. Usually, however, the price „jumps". Graphs illustrating such a situation can be seen on figure 4. In this case two situations were simulated. In the first case, the price was gradually increased. It can be seen that customers' reserves fluctuate and may stabilize in the long term. Reserves on the market rose considerably, which implies that the producer responds sensitively to changes in the selling price. When, after increasing, the price decreased to the initial level, both the customers' reserves and the number of goods on the market adjust correspondingly despite a significant swing in the period after the price change.

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Let us emphasise that the model, describing an economic process more realistically, is again an initial problem for a system of nonlinear differential equations with constant delay Δ, piecewise constant coefficients and piecewise continuous non-homogeneity. Therefore, its solution will be a continuous function and a function piecewise continuously differentiated on interval $[0, \mathrm{T}]$, in other words, a continuous function on $[0, \mathrm{T}]$, and highest possible countable amount of points in the interval, which do not have any derivation.

a) $t \in< 0; 8) \quad p = 0.1$
$t \in< 8; 15) \; p = 0.15$
$t \in< 15; 30) \quad p = 0.2$

$b)$ $t \in< 0; 8) \quad p = 0.15$
$t \in< 8; 15) \; p = 0.2$
$t \in< 15; 30) \quad p = 0.1$

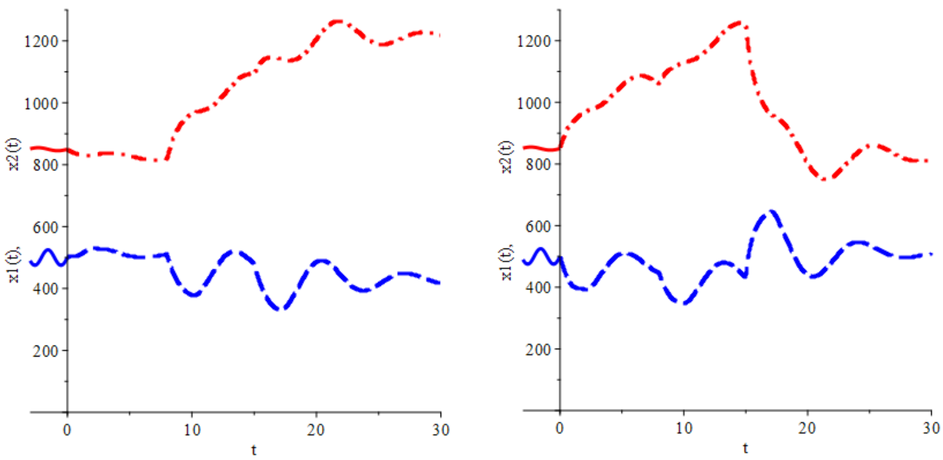

Figure 4. Price „jumps“

6. Conclusion

When modelling complex economic problems, we often face the fact that relations between individual quantities vary in time. One way of including the dynamics of processes in a model is to view time as a continuous quantity even though real observations are not performed continuously. If we allow for this possibility, dynamical models can be described by differential equations. When specifying a model's structure, dynamical character can be grasped by including the delayed effect of exogenous and endogenous variables.

The paper presented a solution to the model of production, warehousing and sale of fast moving consumer goods, represented by the system of nonlinear differential equations with delayed argument. The system has been solved using modern theory and the effect of changes in some parameters on the model's

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solution has been monitored. Theoretical results were supplemented with an illustrative example which brings specific results in graphical form.

In conclusion, it can be observed that the presented way of solving systems of nonlinear equations may significantly help solve particular problems, related to a real company's data, allowing a production plan and pricing strategies to be drawn up with a view to maximizing profit. The solution allows us to evaluate the intensity of the effect of changes in various factors, thus helping to choose the optimum management strategy.

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